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LETTER TO THE EDITOR

Variational approach to the integrals of motion for symplectic maps

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Abstract. A variational approach to compute approximate integrals of motion for symplectic maps is presented. We show that the perturbative solution is recovered when a small neighbourhood of a fixed point is considered. An application to the Hénon map illustrates how a nonlinear resonance is described by this method. A possible extension to the problems of beam dynamics in particles accelerators is discussed.

The transverse oscillations of a particle in an accelerator around its reference orbit (betatronic motion) are usually described by symplectic maps. For a single sextupole magnetic element the map is the familiar quadratic Hénon map [1]; the most general cases are obtained by composing a large number of polynomial maps, thus obtaining a polynomial map of very high order [2]. Such a map contains all the information for the study of a single particle trajectory, i.e. the local properties of the phase space, while in the stability problems of beam dynamics one is rather interested in global properties related to the dynamical aperture of the machine.

Usually the study of the global structure is carried out empirically by direct inspection of a high number of iterations of the map. An alternative, more systematic, approach is given by the perturbative normal form method [3], which provides a description of the orbit in suitable neighbourhoods of an elliptic fixed point, corresponding to the reference central orbit.

In this letter we propose a variational method to extract global information on the topology of phase space by constructing, for a given map, approximate invariants of motion. We work out the variational approach for polynomial invariants, but the extension to more general cases seems straightforward.

When restricting the variational analysis to orbits very close to the origin, one recovers as expected the perturbative results. The variational solution is unique while the perturbative solution is degenerate.

The variational principles played an important role in the development of Lagrangian and Hamiltonian mechanics since Maupertuis and Hamilton's original formulation [4]. Recently the non-existence of uniform quasiperiodic solutions for quasi-integrable Hamiltonian systems led Percival [5] to formulate a fixed frequency variational principle whose solution gives the motion on individual tori. An analogous principle has been formulated by Aubry [6] and Mather [7] in the case of twist area preserving maps. Indeed for a fixed diophantine frequency and for sufficiently small perturbation these variational approaches are equivalent to the usual perturbative method, which can also be used for computational purposes, because the solution is

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analytic and the perturbative series converge. However the variational principle allowed Mather to obtain weak solutions corresponding to orbits whose closure is no longer a torus but a closed Cantor-like subset of a torus (currently referred to as cantorus [8]).

In the case of isochronous Hamiltonian vector fields or diffeomorphisms with a fixed frequency linear part, the perturbative approach consists in conjugating the system with a Birkhoff normal form [9]. It is known that there is in general no analytic solution in a neighbourhood of the origin even though a C^∞ interpolation of the KAM tori [10], which is in some sense a weak solution of our problem, exists. As a consequence the perturbative series diverges but the remainder can be made exponentially small according to the Nekhoroshev theorem [11].

An equivalent, but simpler approach to study the isochronous Hamiltonian vector field consists in the perturbative computation of the integrals of motion. Such a perturbative expansion is, however, expected to be non-convergent because according to a Siegel [12] theorem, a Hamiltonian field with an elliptic fixed point almost surely does not admit analytic integrals of motion beyond the Hamiltonian itself.

In this letter we investigate the possibility of constructing, by a variational principle, approximate solutions to the integrals of motion in a neighbourhood A of an elliptic fixed point when the linear frequency of the map is non-resonant.

The non-resonance condition fixes the germ of the variational solution to be a quadratic form when A is a disk. In this case the non-resonant form perturbation series is recovered when the radius of A tends to zero; a similar result holds for an annulus around a strongly non-resonant torus. When the radius of A is finite the variational solution, which is obtained by solving a linear set of equations, allows to recover non-perturbative structures such as chains of islands. A resonant perturbation theory for symplectic maps [3, 13, 14], which allows to take into account the effects of a nonlinear resonance, is not available for the functional equation of the integrals of motion and requires the solution of a more complicated functional equation and the calculation of an 'interpolating Hamiltonian'. Moreover the variational method can be easily applied to symplectic maps different from polynomial maps and in principle can describe the effects of more than one nonlinear resonance at the same time. Another advantage of the method is its extreme simplicity from the computational point of view, even though some care has to be taken in order to control the conditioning of the system of linear equations, which provides the solution of the problem, when the dimension of the system grows.

Moreover the method is well suited to analyse very complicated maps as in the case of a magnetic lattice, or the results of simulation experiments.

The application of the method to the analysis of the dynamical variables, the geometry of the orbits and the stability properties of the beam dynamics seems to be particularly promising.

Let $\mathcal{M} : \boldsymbol{R}^{2d} \to \boldsymbol{R}^{2d}$ be an analytic symplectic map and A a bounded set of $\boldsymbol{R}^{2d}$; we say that a non-constant real function $I : \boldsymbol{R}^{2d} \to \boldsymbol{R}$ is a first integral of motion for the map $\mathcal{M}$ on A if the following equation is satisfied

$$I \circ \mathcal{M}(\boldsymbol{x}) - I(\boldsymbol{x}) = 0 \qquad \forall \boldsymbol{x} \in A. \tag{1}$$

By defining

$$\mathcal{F}(I) = \int_A (I \circ \mathcal{M} - I)^2 \, \mathrm{d}\mu \tag{2}$$

where $\mathrm{d}\mu$ is the usual Lebesgue measure (which is also the invariant measure since

$\mathcal{M}$ is symplectic), equation (1) can be derived from the variational principle

$$\delta\mathcal{F}(I)=0. \tag{3}$$

As is well known from a theorem due to Siegel [12], if A is an open set, equation (1) does not admit any analytic solution I for a generic analytic map $\mathcal{M}$. Therefore we look for a minimum of the functional (2) to be regarded as an approximate solution of (1).

In order to avoid the trivial (constant almost everywhere) solutions we write I according to

$$I(x)=I_0(x)+\hat{I}(x) \tag{4}$$

where $I_0(x)$ is an explicitly known approximation to $I(x)$ and $\hat{I}(x)$ belongs to a subspace linearly independent from $I_0(x)$. If

$$\mathcal{M}=\mathcal{M}_0\circ\hat{\mathcal{M}} \tag{5}$$

where $\mathcal{M}_0$ is a linear map (with non-degenerate eigenvalues), then the natural choice for I_0 is any one of the d quadratic forms providing the integrals of motion of $\mathcal{M}_0$ and $\hat{I}$ can be chosen in the subspace of the polynomials of order larger or equal to 3.

In the case of the Hénon map of the plane ($d=1$), letting $x=(q,p)$ we have

$$\begin{pmatrix} q' \\ p' \end{pmatrix}=R(\omega)\begin{pmatrix} q \\ p+q^2 \end{pmatrix} \tag{6}$$

where $R(\omega)$ is the rotation matrix of an angle ω, $\mathcal{M}_0$ corresponds to $R(\omega)$ and the quadratic invariant I_0 is given by

$$I_0(x)=p^2+q^2. \tag{7}$$

In order to get approximate solutions we choose $\hat{I}(x)$ in the subspace $\mathcal{P}_N$ of the polynomials of degree n with $3\leqslant n\leqslant N$. Let $\{f_k\}_{k\geqslant 1}$ be the monomial base of $\mathcal{P}_\infty$ (the space of all polynomials of degree $\geqslant 3$), ordered in such a way that the degree of f_k is a non-decreasing sequence. Expanding $I(x)$ in the monomial base we introduce a $\nu(N)$-dimensional vector a, where $\nu(N)$ is the dimension of $\mathcal{P}_N$, according to

$$I(x)=I_0(x)+\sum_{k\geqslant 1}^{\nu(N)} a_k f_k(x). \tag{8}$$

One easily finds that $\nu(N)=\binom{2d+N}{2d}-\binom{2d+2}{2d}$. If $\mathcal{M}$ is a polynomial map of order L, we introduce the rectangular $\nu(LN)\times\nu(N)$ matrix D and the $\nu(LN)$-dimensional vector b defined according to

$$f_k(M(x))=\sum_{j\geqslant k} D_{jk}f_j(x) \tag{9}$$

and

$$I_0(\mathcal{M}(x))-I_0(x)=\sum_{k\geqslant 1} b_k f_k(x). \tag{10}$$

In the case of the Hénon map, for which $L=2$, D is a matrix of order $\nu(2N)\times\nu(N)$ and the $\nu(2N)$-dimensional vector b has only the first $\nu(4)$ entries not equal to zero. With the above notation the functional (2) reads

$$\mathcal{F}(I)=\tilde{a}(\tilde{D}-\tilde{1})G(D-1)a+\tilde{a}(\tilde{D}-\tilde{1})Gb+\tilde{b}G(D-1)a+\tilde{b}Gb \tag{11}$$

where $\mathbf{1}$ is a rectangular matrix $\nu(LN)\times\nu(N)$ whose first $\nu(N)$ rows are the $\nu(N)\times\nu(N)$ identity matrix, and the remaining rows are all zero, and G is the Grassmann matrix

$$G_{ij}=\int_A f_i(x)f_j(x)\,\mathrm{d}\mu(x). \tag{12}$$

The minimum of $\mathscr{F}(I)$ is given by the solution of the linear system

$$(\tilde{D}-\tilde{\mathbf{1}})G(D-\mathbf{1})a=-(\tilde{D}-\tilde{\mathbf{1}})Gb. \tag{13}$$

The matrix $(\tilde{D}-\tilde{\mathbf{1}})G(D-\mathbf{1})$ is non-singular since the vectors of the null space of this matrix would provide exact integrals of motion as polynomials of finite order N. If the initial map $\mathcal{M}$ is non-integrable (generic case) this is clearly impossible.

In order to discuss the connection with the perturbative approach we first observe that equation (1), written in the polynomial base, reads

$$(D-1)a=b. \tag{14}$$

Choosing the phase space coordinates so that $\mathcal{M}_0$ is diagonal, then the $\nu(N)$-diagonal elements of $(D-\mathbf{1})$ are given by $\lambda_1^{m_1}\ldots\lambda_{2d}^{m_{2d}}-1$, $|m|\leqslant N$, where λ_j are the eigenvalues of $\mathcal{M}_0$. As a consequence the elements corresponding to the normal form monomials $\{x^m|\lambda_1^{m_1}\ldots\lambda_{2d}^{m_{2d}}-1=0\}$ vanish.

The perturbative method consists in projecting into the $\nu(N)$ first components of our polynomial base

$$\Pi_N(D-\mathbf{1})a=\Pi_N b \tag{15}$$

where

$$\Pi_N=\begin{pmatrix} G_0 & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{pmatrix} \tag{16}$$

and G_0 is any invertible matrix, in particular the reduced $\nu(N)\times\nu(N)$ Grassmann matrix. The presence of normal form monomials reduces the rank of the linear system (15) to $r<\nu(N)$ and the solution depends on $\nu(N)-r$ arbitrary constants. That corresponds to the fact that if I is a perturbative solution then

$$I'=I_0+g(I) \tag{17}$$

where $g(I)$ is a polynomial of order 2 at least, is also a solution.

When A is a disk of radius r, in the limit $r\to 0$ the Grassmann matrix G approximates Π_N with an error of order r^{N+1} so that we recover the perturbative solution exactly and the system (13) becomes equivalent to the system (15). Moreover the degeneracy of the perturbative solution implies that the limit $r\to 0$ is singular. But if we are in a more general situation and the perturbative approach cannot be used, the system (10) which takes into account the contribution of all the terms in the expansion of $f_k\circ\mathcal{M}$, still provides the best approximate integral of motion in the subspace $\mathscr{P}_N$. Moreover we recall that the perturbative method in a neighbourhood of an invariant torus requires a particular coordinates system (ρ,θ) where ρ measures the 'distance' from the torus and we need to know the parametric equation of the torus to perform a non-trivial change of coordinates. This procedure is very cumbersome from a numerical point of view. In contrast, the variational approach requires only a numerical knowledge of the invariant torus and works in every base system $\{f_k\}$. However as the numerical results suggest, in a general case the matrix (17) is a full, symmetric, ill-conditioned matrix (the ratio between the biggest and the smallest eigenvalue is $\gg 1$) so that solving the linear system (9) is not a trivial problem. In order to overcome such a difficulty one should partition the phase space into a large number of small annuli and/or use preconditioning techniques.

A rigorous estimate of the error $I \circ \mathcal{M} - I$ is not yet available but the results of the perturbative theory suggest that a Nekhoroshev-type estimate may still hold for systems which can be locally approximated by integrable systems. However the answer to the question of how accurate can the interpolation of the orbits of a symplectic map $\mathcal{M}$ by means of the level sufaces of set of first integrals of motion be, may be connected with the existence of weak solutions for the variational principle (3) which has not yet been studied.

We have used the variational method for computing an approximate integral of motion for the Hénon map (6) in a subset A of the stability region of the elliptic fixed point. The linear frequency $\omega/2\pi$ was fixed at the value 0.21 so that a nonlinear resonance of order 5 causes the presence of a chain of 5 islands (see figure 1) near the border of the stability basin of the origin.

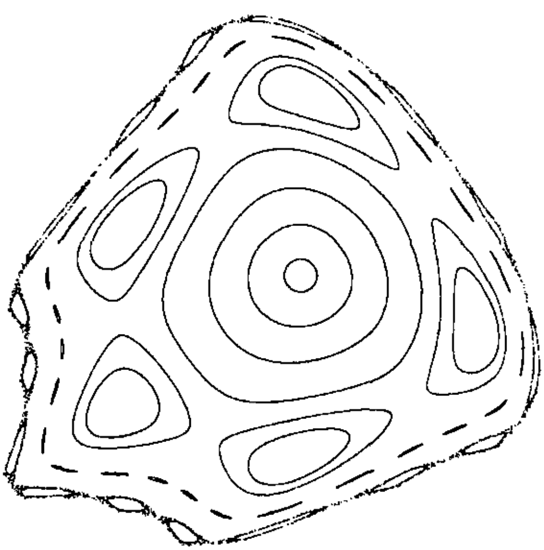

Figure 1. Phase plot of the Hénon map obtained by directly iterating the map; the linear frequency is chosen as $\nu = 0.21$.

Our aim is to reproduce accurately the geometry of the orbits in such a basin. In this case the non-resonant perturbative approach to the integrals of motion, which describes only closed curves, fails completely when the chain of islands is approached. In this case the chain of islands can be described by the resonant perturbation theory based on the resonant Birkhoff normal forms [3, 13, 14], even though the accuracy is limited when the chain is very close to the dynamical aperture where the perturbation parameter has a large value. However the variational approach allows us, in principle, to describe any number of topological structures such as chains of an arbitrary number of islands and islands within islands, whereas the resonant perturbation theory can only reproduce closes curves and chains of islands whose order is a multiple of the resonance order. Moreover the resonant perturbation approach has been developed for polynomial maps whereas the variational method can work for any base system of functions.

An exhaustive comparison between the variational method and the perturbative method will be the object of a future work.

We look for a polynomial integral of motion such as (8) where the initial condition $I_0(x)$ is chosen according to (7). The entries of the matrix $(\tilde{D}-\tilde{1})G(D-1)$ and of the known term $(\tilde{D}-\tilde{1})Gb$, see equation (13), are given by the integrals

$$\int_A (\bar{I}_h \circ \mathcal{M} - \bar{I}_h)(\bar{I}_k \circ \mathcal{M} - \bar{I}_k)\,\mathrm{d}\mu$$
$$\int_A (I_k \circ \mathcal{M} - I_k)(I_0 \circ \mathcal{M} - I_0)\,\mathrm{d}\mu. \tag{18}$$

In order to simulate a realistic situation we have only assumed a numerical knowledge of the map $\mathcal{M}$ and have therefore replaced the integrals (18) with the sums

$$\sum_{j=1}^{m} (I_h(x_j') - I_h(x_j))(I_k(x_j') - I_k(x_j))$$
$$\sum_{j=1}^{m} (I_k(x_j') - I_k(x_j))(I_0(x_j') - I_0(x_j)) \tag{19}$$

where $\{x_j\}$ is a distribution of m points in the set A and $x_j' = \mathcal{M}(x_j)$. We have not used any optimization technique in the choice of the distribution $\{x_j\}$ but we have checked that the results are stable when we change the sampling. Fixing the highest degree of the polynomials in the expansion (8) to $N=9$ a uniform distribution of 2000 points turns out to be sufficient.

In figure 2 we have plotted the level curves $I(x) = \text{constant}$ of our approximate integral of motion; they are to be compared with the orbits in figure 1 which are the results of a direct tracking obtained by iterating the map. The scale of the two figures is the same. Note the quite good agreement between the orbits and the level curves;

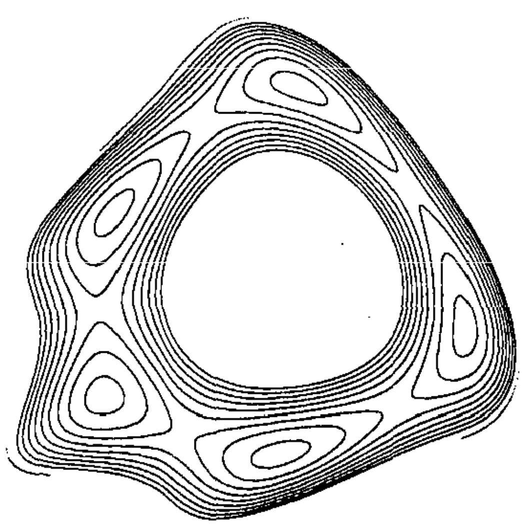

Figure 2. Level curves of the integral motion provided by the variational method; the scale of both figure 1 and figure 2 is the same.

the integral of motion properly reproduces the two different topological structures: the closed orbits around the origin and the island structure at larger distance. The information contained in the tracking data is equivalent to the information provided by the variational integral of motion. It is clear that the same method can be used starting from the tracking data which describe the motion of a charged particle in the magnetic lattice of a hadron accelerator. The method requires as input a relatively small number of tracking data; its further advantages are the information compression into a small number of parameters and a global description of the geometry of the orbits, well suited for graphical representation, which allows us to draw a conclusion on the long-time behaviour of the orbits and on their stability properties.

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